# Everything you need to know about integration applications

This section will have some ways of integrating functions, as well as a revised mean value theorem and how to deal with unbounded intervals and functions this document does not include how to deal with area computation (I’ll consider this intuitive), work and mass, arclength, area calculation using arclength, probability, and special functions (gamma function , error function , and others ).

I will cover partial fraction even though it wasn’t in MAT137 for the 2017-18 year. I will only cover the basics.

## Anti-derivatives to memorise

This will be a table of integrals or “anti-derivatives” that you should know. If you want more, look at the EYNTKA derivatives sheet at the “derivatives to memorise”

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## u-substitution

you know u-substitution, so I’ll go over the formulas rather quickly

I’ll run over a bunch of tricks that’d be useful in integrating:

HERE

## Integration by parts

You know integration by parts, so I’ll go over the formulas rather quickly

I’ll run over a bunch of tricks that’d be useful in integrating:

## Partial fraction integration

HERE

## Improper integrals

We want to be able to integrate integrals with unbounded intervals or unbounded functions. Since they’re not “proper” in the sense that the idea of Darboux sums becomes hard to understand and so they’re “improper”. We’re going to start with unbounded intervals and then unbounded functions.

### Unbounded intervals

Definition: if is an integrable function, the improper integral of an is the limit:

We say the improper integral converges if this limit is finite, and diverges otherwise.

If an the domain converges on the bottom, then the limit approaches . If both the top and bottom tend towards infinity, then we must be careful on how we define our integrand. Simply doing

Won’t suffice, because some function will have area that constantly cancels out (like ), but this function doesn’t converge in the sense we want, that is it shouldn’t depend on the speed of converges on the left and right side. We want it to produce the same value as

This wouldn’t be the case for sin(x), which would break down at this point. To get around this, we’ll define:

We don’t need to check . That was just me making sure that the idea that choosing any starting point shouldn’t effect our value. In practice, you should think of .

You know how to compute limits, and if you don’t remember look at EYTNKA limits.

#### Evaluating improper integrals with unbounded intervals

Example: determine

Solution:

Example: determine if exsits

Notice that the limit of the anti-derivative doesn’t exists, therefore the limit does not exist

Example: determine

Solution:

Example: determine

This function diverges, therefore, there’s no value.

Example: determine

Solution: We need to choose a value at which we could split this function. 0 is an easy candidate and would do the job very well. I skipped writing the limits to save myself time:

When t is negative

Therefore:

Example: determine

Solution:

Therefore:

#### Comparison Tests

Sometimes, we don’t want to determine an integral’s exact value, but whether an integral converges or not. For example:

There aren’t any elementary solutions to this integral. Therefore, you’ll have a few test to check whether an integral converges.

##### p-test

The first test is called the p-test. It only works for functions with a single monomial in the denominator.

Definition: if , and is integrable, then

Converges if and only if

Proof: it’s in the book. It’s quite easy (p=1 then log(x), if p not 1 then power rule).

Note that any constant coefficient will not effect convergence. This function is sort of a boundary case for convergence. However, be careful applying it instantly, since this is not an exact boundary, but a very close to it. For example:

Converges quicker and , but still diverges.

##### The basic comparison test for unbounded intervals (BCT)

This test compares two values, one being “bigger” than the other

Theorem: suppose and satisfy for all

The contrapositive is also very useful

Proof: not here, in book page 651. The proof uses two lemmas. The first shows that if one function is bounded by another, then if bigger converges, then smaller converge. If smaller diverges, then bigger diverges. Second lemma related bounds to convergence.

Be careful when you do the BCT, if , and f(x) converges, or g(x) diverges, that tell you nothing! the comparisons are usually very easy: you either want to default to something you already know, or a situation where you could use the p-test. If there’s a sin or cosine, then you could bound it and “get rid of it”.

Example: that

Solution: when it comes to sin and cosine, there are easy ways to eliminate by noticing there values are bounded by

For our situation, notice that

Taking that into account, we have

We kown that diverges by the p-test, so by the BCT, this integral also diverges.

Example: Determine whether

Notice that

Which converges by the p-test, and so by the BCT, this converges.

##### The limit comparison test (LCT)

Sometimes, we want a slightly more clever solution. An example were BCT wouldn’t be enough his

the natural comparison

However, this doesn’t tell us anything, the divergence of doesn’t tell us if the left function converges or not. We need to be a little cleverer. In this case

Follows the right requirements for BCT. From this, we know that the integral diverges. This is a simple example were a clever solution could be used, but we could get something along the lines of

Which we wouldn’t be easy with BCT. What we want is a solution in were we could take the first step of doing a BCT (like the in our first example), and solve from there. We’ll use a more developed method of the BCT: instead of comparing two function via inequalities, we’ll see if there quotient can give us any information. Namely, if they both converge or diverge, then we’ll have a value between 0 and infinity, but not one or the other.Note that it doesn’t matter where f(x) is on to or bottom, because by the limit laws, if f(x)/g(x) = L, then g(x)/f(x) = 1/L. (WHAT IF THEY BOTH DIVERGE? And STILL )

Theorem: suppose and are positive integrable functions. If

Notice that if one of them diverges, then the limit condition doesn’t hold since it would be 0 or .

Proof: in book page 653 the proof requires a lemma used for the BCT

Example: determine whether

Solution: following the same logic as that in the introduction of this section:

However, the divergence of x tells us nothing about the value of the left integral. Instead of trying to come up with a clever solution, which in some cases could be really hard, we’ll use the LCT from the information already given, we get that

Therefore, by the LCT, since diverges, so does the other integral.

Example: determine whether converges or diverges

Solution: This is an excellent example were using the BCT is possible but would take more work than the LCT. You know that if you multiply everything by

And then work keep gruelling over till you get a good comparison, you’ll find a value. However, take this intuition and convert it into a use in the LCT. Set and test if

Therefore, since converges, by the LCT, the previous integral converges as well

Example: determine whether converges or diverges

Solution: this one would be a nightmare to do BCT. For crazy terms like this, the trick is to look at the fastest growing function in the nominator and denominator, take the quotient of them, and then do the limit comparisons test. This seems non-sensical, but the purpose is to get constants in the nominator and denominator. Next, if this g(x) diverges, then f(x) diverge. If it converges, then the f(x0 convergesThis process is demonstrated on the bottom. Set, which we know converges (since grows quicker than )

Since g(x) converges, then by the LCT, so does the other integral .

#### Absolute convergence?

Not nessesary for the upcoming test. I’ll be brief: if the absolute value of the sequence converges, the sequence converges. If it doesn’T converge, but converges absolutely, then it is said to converge conditionally.

### Unbounded functions

Notice that is unbounded around x=0. This is important, because trying to compute an integral in the following fashion:

Is completely erroneous. A HUGE red-flag is that the function is POSITIVE and should yield a positive value. Since the function is unbounded, the fundamental theorem of calculus does not apply. The trick is to approach the value “epsilon close”.

Definition: if is unbounded at b but integrable on for every , the improper integral of f is

Similarly for the reverse:

If we want to integrate from both sides, then do the left and the then the right, just like for unbounded integrals. If the limit is finite, we say that the improper integral converges; otherwise we say that the improper integral diverges. This is once again introducing limit to solve the problem with limits. Let’s take the previous integral, but only look at one side (as the integral will show)

Therefore, the function diverges. I’ll cover some other examples here

#### Evaluating improper intervals with unbounded functions

Example: determine , if it converges.

Solution: the problem arises at x=2. Therefor

Example , if it exists

Evaluate , if it converges

#### Comparison tests

Like unbounded intervals, unbounded functions have convergence tests. I’ll briefly cover them:

##### The p-test

Theorem: for any a>0, then integral converges if and only if

Proof: if p=1, I’ll let it be intuitive that ln(x) ->

If

Notice that the p-test for unbounded intervals has the property that p > 1, while for unbounded function, it is the opposite: p < 1, since we’re approaching zero instead of infinity! That’s a very important distinction! Q.E.D.

I won’t do example here, but I’ll incorporate in the BCT for unbounded function

##### The basic comparison test for unbonded functions

The intuition is identical to that of BCT. I’ll name it for the sake of formality. I’ll also include the two lemmas that are needed for the proof. This is because of their use in sequences and series

Lemma 1:

Let be two functions such that F is unbounded at the point b. if G F. for all x near b, then G is also unbounded at b; that is:

Proof: here

Lemma 2:

There a very important lemma that becomes vital to sequences and series is :

Proof: here

Theorem: let be unbounded at b and integrable on for every . Suppose as well that for all then:

The contrapositive is also very useful

Examples:

##### Limit comparison test (LCT)

Identical to LCT for unbounded intervals

Theorem: let be non-negative functions, unbounded at b and integrable on for all if

Then:

Example:

## MVT for integrals

The MVT for integrals creates a constant, such that would create a rectangle with the area of the integrated function.

Ex: determine the average value of on

Solution:

INTUITION HERE

### the first mean value theorem for integrals

This one simply states that intersects the function somewhere.

if is continuous with mean value , there exists such that

### The second mean value theorem for integrals

For this one, I’ll explain after the proof

if is continuous g is integrable, non-negative function, there exists such that

Proof: For this proof, take it part by part. You apply properties that are given, and notice that we’re working in bounded intervals; therefore existence theorems could and will be used.

Since f is continuous on , by the EVT, it achieves it maximum and minimum on ], which we’ll call m and

M. we know that g(x) is non-negative, so

Since the integral are monotone, this imlies that

If is zero then the proof is done. If is not zero then we could divide like so:

Since is continuous by IVT, :

If , then the second last representation becomes the original mean value theorem. In a way, this is a fancier representation if there are two function’s at work.

### The third mean value theorem for integrals

if is continuous g is monotone (increasing or decreasing), then there exists such that

The proof is difficult; tyler didn’t even put the full thing in the book. I don’t fully understand it yet. (p.596)

## Useful Tricks

In this section, I’ll be listing a bunch of tricks that are useful in solving integrals. This is a short list covering only the ones that could be presented in MAT135/137/157[[1]](#footnote-1). For a more complete list of integrating technics involving tricks usually seen in integration bees, look at EYTNKA integration tricks.

1. when solving an integral like this one

Notice that taking continuous differentiation of will constantly shrink it to smaller degrees, while cos(x) will simply oscillate. This makes integrating by parts super useful:

1. Let’s say your given the following integral

You remember that if the denominator’s constant and quadratic were 1, this is simply tan(x).To solve this, do some clever u-sub. First I’ll generalise then I’ll do the specific

The rest you could do yourself. **DON’T FORGET C!!!!**

1. To integrate inverse function, take advantage of this fact:

So

1. You could manipulate your u in and out of hard functions like sin or roots. To show what I mean, here’s an example:

Notice:

Therefore:

From here on out the integral is easy to solve.

1. Because don’t want to fall behind the other math classes now 😊 [↑](#footnote-ref-1)